COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA 29 (2) (2009) 227–262
DOI: 10.1007/s00493-009-2409-2

RAMSEY GOODNESS AND BEYOND

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Received June 6, 2007
Revised January 6, 2008

In a seminal paper from 1983, Burr and Erdős started the systematic study of Ramsey numbers of cliques vs. large sparse graphs, raising a number of problems. In this paper we develop a new approach to such Ramsey problems using a mix of the Szemerédi regularity lemma, embedding of sparse graphs, Turán type stability, and other structural results. We give exact Ramsey numbers for various classes of graphs, solving five – all but one – of the Burr–Erdős problems.

1. Introduction

Our notation is standard (e.g., see [2]). In particular, $G(n)$ stands for a graph of order n; we write $|G|$ for the order of a graph G and $k_r(G)$ for the number of its r-cliques. The join of the graphs G and H is denoted by $G+H$.

Given a graph G, a 2-*coloring* of $E(G)$ is a partition $E(G)=E(R)\cup E(B)$, where R and B are graphs with $V(R)=V(B)=V(G)$. The *Ramsey number* $r(H_1,H_2)$ is the least number n such that for every 2-coloring $E(K_n)=E(R)\cup E(B)$, either $H_1\subset R$ or $H_2\subset B$.

The aim of this paper is to develop a new approach to Ramsey numbers of cliques vs. large sparse graphs. We prove a generic Ramsey result about certain classes of graphs, thus producing an unlimited source of specific exact Ramsey numbers. This enables us to answer a number of open questions and extend a substantial amount of earlier research. Moreover, some of the auxiliary results used in our proofs may be regarded as general tools for wider classes of Ramsey problems.

Mathematics Subject Classification (2000): 05C55; 05C35

Let us recall the notion of goodness in Ramsey theory, introduced by Burr [8]: a connected graph H is p-good if the Ramsey number $r(K_p, H)$ is given by

$$r(K_p, H) = (p-1)(|H|-1)+1.$$

The systematic study of good Ramsey results was initiated by Burr and Erdős in [7]; for surveys of subsequent progress the reader is referred to [10] and [15].

First we outline some of the problems raised in [7].

1.1. Solved and unsolved problems about p-good graphs

In [7] Burr and Erdős, probing the limits of p-goodness, gave some general constructions of p-good graphs and raised a number of questions, most of which are still open. To state the most important problem raised in [7] and reiterated in [10] and [11], we recall that a graph is called *q-degenerate* if each of its subgraphs contains a vertex of degree at most q.

Conjecture 1.1. For fixed $q \geq 1$, $p \geq 3$, all sufficiently large q-degenerate graphs are p-good.

A weaker version of this conjecture was stated earlier by Burr in [8].

Conjecture 1.2. For fixed $q \geq 1$, $p \geq 3$, all sufficiently large graphs of maximum degree at most q are p-good.

Brandt [4] showed that for $p = 3$ and $q \geq 168$, every q-regular graph of sufficiently large order and with sufficiently large expansion factor is a counterexample to Conjecture 1.2. In [27] we showed that, for $p=3$, almost all 100-regular graphs are counterexamples to Conjecture 1.2, and thus to Conjecture 1.1.

We shall answer in the affirmative all but one of the remaining questions raised in [7].

Write C_n for the cycle of order n. Burr and Erdős [7] showed that the wheel K_1+C_n is 3-good for $n \geq 5$. This result motivated the following three questions ([7], p. 50).

Question 1.3. Is the wheel K_1+C_n p-good for fixed $p>3$ and n large?

Recall that the kth power of a graph G is a graph G^k with $V(G^k)=V(G)$ and $uv \in E(G^k)$ if u and v can be joined in G by a path of length at most k.

Question 1.4. Is $K_1+C_n^k$ a p-good graph for fixed $k \geq 2$, $p \geq 3$ and n large?

Question 1.5. Fix $p \geq 3$, $l \geq 1$, $k \geq 1$, and a connected graph G. Is it true that, for every large enough graph G_1 homeomorphic to G, the graph $K_l + G_1^k$ is p-good?

Burr and Erdős estimated that finding an answer to Question 1.5 would be very difficult. In this paper we answer Question 1.5 in the affirmative, implying an affirmative answer to Questions 1.3 and 1.4 as well.

Clearly, the clique number of p-good graphs must grow rather slowly with their order. Therefore, the following question comes naturally ([7], p. 41).

Question 1.6. Subdivide each edge of K_n by one vertex. Is the resulting graph p-good for p fixed and n large?

Burr and Erdős also asked a question about tree-like constructions of fixed families of graphs, which they called "graphs with bridges" ([7], p. 44). We restate their question in a much stronger form.

Given a graph G of order n and a vector of positive integers $\mathbf{k} = (k_1, \ldots, k_n)$, write $G^{\mathbf{k}}$ for the graph obtained from G by replacing each vertex $i \in [n]$ with a clique of order k_i and every edge $ij \in E(G)$ with a complete bipartite graph K_{k_i,k_j}.

Question 1.7. Suppose $K \geq 1$, $p \geq 3$, T_n is a tree of order n, and $\mathbf{k} = (k_1, \ldots, k_n)$ is a vector of integers with $0 < k_i \leq K$ for all $i \in [n]$. Is $T_n^{\mathbf{k}}$ p-good for n large?

We shall answer Questions 1.6 and 1.7 in the affirmative. However, the following particular question raised in [7] is beyond the scope of this paper.

Question 1.8. Is the n-cube 3-good for n large?

1.2. Some highlights on Ramsey goodness

We list below several important results on Ramsey goodness.

Define a *q-book* of size n to be the graph $B_q(n) = K_q + nK_1$, i.e., $B_q(n)$ consists of n distinct $(q+1)$-cliques sharing a q-clique.

Fact 1.9 **([24]).** *For fixed $q \geq 2$, $p \geq 3$, and large n,*

$$r(K_p, B_q(n)) = (p-1)(n+q-1)+1.$$

In the following results K_p is replaced by a supergraph $H \supset K_p$ such that $r(H,G) = r(K_p, G)$ for certain p-good graphs G.

Fact 1.10 (**[13, 16]**). *For fixed $m \geq 1$ and large n,*

$$r(B_2(m), C_n) = 2(n-1)+1.$$

Fact 1.11 (**[29, 17]**). *For fixed $p \geq 2$, $m \geq 1$, and any tree T_n of large order n,*

$$r(B_p(m), T_n) = p(n-1)+1.$$

Write $K_p(t_1, \ldots, t_p)$ for the complete p-partite graph with part sizes $t_1, \ldots, t_p$ and set $K_p(t) = K_p(t, \ldots, t)$.

Fact 1.12 (**[6, 12]**). *For fixed $m \geq 1$, $k \geq 1$, $n_1, \ldots, n_k$, and n large,*

$$r\left(B_2(m), K_{k+1}\left(n_1, \ldots, n_k, n\right)\right) = 2\left(n_1 + \cdots + n_k + n\right) - 1.$$

Fact 1.13 (**[5, 12]**). *For fixed $p \geq 2$, $t \geq 1$, and any tree T_n of large order n,*

$$r\left(K_{p+1}\left(1, 1, t, \ldots, t\right), T_n\right) = p(n-1)+1.$$

Fact 1.14 (**[28, 14, 25, 26]**). *There exists $c > 0$ independent of n such that if n is large and $m \leq cn$, then*

$$r(B_2(m), B_2(n)) = 2n+3.$$

The following result answers in the affirmative a special case of Question 1.7.

Fact 1.15 (**[21]**). *For fixed $p \geq 3$ and graph H, the graph $K_1 + nH$ is p-good for n large.*

2. Main results

We first outline the approach to Ramsey numbers adopted in this paper.

For every p and n, we describe two families of graphs $\mathcal{R}(n)$ and $\mathcal{B}(n)$ such that, if n is large, then for every 2-coloring $E(K_{p(n-1)+1}) = E(R) \cup E(B)$, either $H \subset R$ for some $H \in \mathcal{R}(n)$ or $G \subset B$ for all $G \in \mathcal{B}(n)$.

To describe $\mathcal{R}(n)$, we define joints: call the union of t distinct p-cliques sharing an edge a *p-joint* of size t; denote the maximum size of a p-joint in a graph G by $js_p(G)$. The family $\mathcal{R}(n)$ consists of all $(p+1)$-joints of size at least cn^{p-1} for some appropriate $c > 0$.

To describe $\mathcal{B}(n)$, we first define splittable graphs: given real numbers $\gamma, \eta > 0$, we say that a graph $G = G(n)$ is *(γ, η)-splittable* if there exists a set $S \subset V(G)$ with $|S| < n^{1-\gamma}$ such that the order of any component of $G - S$ is at most ηn. The family $\mathcal{B}(n)$ consists of all q-degenerate (γ, η)-splittable graphs, where q and γ are fixed and $\eta > 0$ is appropriately chosen.

2.1. The main theorem

Here is our main theorem.

Theorem 2.1. *For all* $p \geq 3$, $q \geq 1$, $0 < \gamma < 1$, *there exist* $c > 0$, $\eta > 0$ *such that if* n *is large and* $E(K_{p(n-1)+1}) = E(R) \cup E(B)$ *is a* 2*-coloring, then one of the following statements holds:*

(i) R *contains a* $(p+1)$*-joint of size* cn^{p-1}*;*

(ii) B *contains every* q*-degenerate* (γ, η)*-splittable graph* G *of order* n*.*

Note that Theorem 2.1 gives exact Ramsey numbers for graphs of varying structure, implying, in particular, positive answers to the questions raised in Section 1.1.

2.2. Variations of the $\mathcal{R}(n)$ family

The condition $js_{p+1}(R) > cn^{p-1}$ implies the existence of various $(p+1)$-partite graphs in R. On the one hand, R contains dense supergraphs of K_{p+1} as shown in the following theorem, proved in Section 3.2.

Theorem 2.2. *For all* $p \geq 3$, $q \geq 1$, $0 < \gamma < 1$, *there exist* $c > 0$, $\eta > 0$ *such that if* n *is large and* $E(K_{p(n-1)+1}) = E(R) \cup E(B)$ *is a* 2*-coloring, then one of the following statements holds:*

(i) R *contains* $K_{p+1}(1, 1, t, \ldots, t)$ *for* $t = \lceil c \log n \rceil$*;*

(ii) B *contains every* q*-degenerate,* (γ, η)*-splittable graph* G *of order* n*.*

Observe that this theorem considerably changes the usual setup for goodness results: now in the graph R we find dense supergraphs of K_{p+1} whose order grows with n. On the other hand, if we give up density, we find in R sparse $(p+1)$-partite graphs whose order is linear in n. More precisely, we have the following theorem, proved in Section 3.3.

Theorem 2.3. *For all* $p \geq 2$, $q \geq 1$, $d \geq 2$, $0 < \gamma < 1$, *there exist* $\alpha > 0$, $c > 0$, $\eta > 0$ *such that if* n *is large and* $E(K_{p(n-1)+1}) = E(R) \cup E(B)$ *is a* 2*-coloring, then one of the following statements holds:*

(i) R *contains* $K_s + H$ *for every* $(p+1-s)$*-partite graph* H *with* $|H| = \lfloor \alpha n \rfloor$ *and* $\Delta(H) \leq d$*;*

(ii) B *contains every* q*-degenerate,* (γ, η)*-splittable graph* G *of order* n*.*

2.3. Variations of the $\mathcal{B}(n)$ family

Call a family of graphs $\mathcal{F}$ *γ-crumbling*, if for any $\eta > 0$, there exists $n_0(\eta)$ such that all graphs $G \in \mathcal{F}$ with $|G| > n_0(\eta)$ are (γ, η)-splittable. We will say that a family $\mathcal{F}$ is degenerate and crumbling if $\mathcal{F}$ consists of q-degenerate and γ-crumbling graphs for some fixed q and γ.

Restricting Theorem 2.1 to degenerate crumbling families, we obtain the following theorem.

Theorem 2.4. *For all $p \geq 2$, $q \geq 1$, $0 < \gamma < 1$ there exists $c > 0$ such that if $\mathcal{F}$ is a q-degenerate γ-crumbling family, n is large, and $E(K_{p(n-1)+1}) = E(R) \cup E(B)$ is a 2-coloring, then one of the following statements holds:*
(i) R contains a $(p+1)$-joint of size cn^{p-1};
(ii) B contains every $G \in \mathcal{F}$ of order n.

Since K_{p+1} is a subgraph of any $(p+1)$-joint, it follows that all sufficiently large members of a degenerate crumbling family are p-good. This simple observation is a clue to the answers of all questions of Section 1.1.

Subdivide each edge of K_n by a single vertex, write $\widehat{K_n}$ for the resulting graph, and note that $\widehat{K_n}$ is 2-degenerate. If we remove the vertices of the original K_n, the remaining graph consists of $\binom{n}{2}$ isolated vertices. Since $n < (n(n+1))^{1/2}$ it follows that $\widehat{K_n}$ is $(1/2, \eta)$-splittable for $\eta = 1/\binom{n+1}{2}$. Thus, the family of all $\widehat{K_n}$'s is 2-degenerate and crumbling; hence, $\widehat{K_n}$ is p-good for n large, answering Question 1.6.

The propositions stated below are proved in Section 4 unless their proof is omitted.

The answer to Question 1.5 is affirmative in view of the following three propositions.

Proposition 2.5. *The family of all graphs homeomorphic to a fixed connected graph G is degenerate and crumbling.*

Proposition 2.6. *If $\mathcal{F}$ is a crumbling family of bounded maximum degree, then, for fixed $k \geq 1$, the family $\mathcal{F}^k = \{G^k : G \in \mathcal{F}\}$ is degenerate and crumbling.*

The following proposition is obvious, so we omit its proof.

Proposition 2.7. *Let $l \geq l$ be a fixed integer. If $\mathcal{F}$ is a degenerate crumbling family, then the family of connected graphs $\mathcal{F}^* = \{K_l + G : G \in \mathcal{F}\}$ is degenerate and crumbling.* ∎

Note also that, in view of Proposition 2.7, Theorem 2.4 generalizes Fact 1.15.

Trees provide various examples of degenerate crumbling families.

Proposition 2.8. *Every infinite family of trees is degenerate and crumbling.*

In particular, Proposition 2.8 and Theorem 2.4 extend Fact 1.13. Likewise, Theorem 2.4 and the following simple observation, whose proof we omit, extend Fact 1.9.

Proposition 2.9. *Every infinite family of q-books is degenerate and crumbling.* ∎

Some operations on graphs fit well with degenerate and crumbling families, as shown in Proposition 2.6 and the following two propositions.

Proposition 2.10. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be degenerate crumbling families. Then the family*

$$\mathcal{F}_1 \times \mathcal{F}_2 = \{G_1 \times G_2 \colon G_1 \in \mathcal{F}_1,\, G_2 \in \mathcal{F}_2\}$$

is degenerate and crumbling.

Proposition 2.11. *Let $\mathcal{F}$ be a degenerate crumbling family, and $\{\mathbf{k}_n = (k_1, \ldots, k_n)\}_{n=1}^{\infty}$ be a sequence of integer vectors with $0 < k_i \leq K$, for $i \in [n]$. Then the family $\mathcal{F}^* = \{G^{\mathbf{k}_n} : G \in \mathcal{F},\, |G| = n\}$ is degenerate and crumbling.*

Note that Proposition 2.11, together with Theorem 2.4 answers Question 1.7 in the affirmative.

As an additional application consider the following example: write $Grid_n^k$ for the product of k copies of the path P_n, i.e., $V(Grid_n^k) = [n]^k$ and two vertices $(u_1, \ldots, u_k)$, $(v_1, \ldots, v_k) \in [n]^k$ are joined if $\sum_{i=1}^{k} |u_i - v_i| = 1$. Propositions 2.8, 2.10, and Theorem 2.4 imply that $Grid_n^k$ is p-good for k fixed and n large; it seems that this natural problem hasn't been raised earlier.

A particular instance of Theorem 2.4 is the following extension of Facts 1.10, 1.11, 1.12, and 1.14.

Theorem 2.12. *For all $p \geq 2$, $q \geq 1$, $\gamma > 0$ there exist $c > 0$ such that for every q-degenerate γ-crumbling family $\mathcal{F}$ of connected graphs, then*

$$r\left(B_p\left(\lceil cn \rceil\right), G\right) = p(n-1) + 1$$

for every $G \in \mathcal{F}$ of sufficiently large order n.

Indeed, it suffices to note that if $js_{p+1}(R) > cn^{p-1}$, then $B_p(\lceil cn \rceil) \subset R$.

Restricting Theorem 2.3 to crumbling degenerate families, we can substantially generalize Theorem 2.12 and replace the graph $B_p(\lceil cn \rceil)$ with other graphs, e.g., $K_{p-1} + C_{\lceil cn \rceil}$, where $\lceil cn \rceil$ is even.

2.4. Remarks on the proof methods

The proof of Theorem 2.1 is based on several major results. The key element is a compound of the Szemerédi regularity lemma and a structural theorem in [22], stating that, for sufficiently small $c>0$, the vertices of any graph G with $k_p(G)<cn^p$ can be partitioned into bounded number of very sparse sets. Other ingredients are a stability result about large p-joints, proved in [3], and a probabilistic lemma used in different forms by other researchers. Finally we construct several rather involved embedding algorithms for degenerate splittable graphs.

3. Proofs

We start with some additional notation.

Additional notation

Write:

- $[n]$ for the set $\{1,\ldots,n\}$ and $[n..m]$ for the set $\{n,n+1,\ldots,m\}$;
- $X^{(k)}$ for the collection of k-element subsets of a set X.

Given a graph G write:

- $\delta(G)$ and $\Delta(G)$ for the minimum and maximum degrees of G;
- $\omega(G)$ for the clique number of G;
- $\psi(G)$ for the order of the largest component of G.

Given disjoint nonempty sets $X,Y\subset V(G)$, write:

- $e_G(X,Y)$ for the number of $X-Y$ edges and $\sigma_G(X,Y)$ for $e_G(X,Y)/(|X||Y|)$;
- $e_G(X)$ for the number of edges induced by X and $\sigma_G(X)$ for $2e_G(X)/|X|^2$;
- $G[X]$ for the graph induced by X;
- $\Gamma_G(X)$ for the set of vertices joined to all $u\in X$ and $d_G(X)$ for $|\Gamma_G(X)|$.

In any of the functions $e_G(X,Y)$, $\sigma_G(X,Y)$, $\sigma_G(X)$, $e_G(X)$, $\Gamma_G(X)$, and $d_G(X)$ we drop the subscript if the graph G is understood.

Given $\varepsilon>0$, a pair (A,B) of nonempty disjoint sets $A,B\subset V(G)$ is called *ε-regular* if $|\sigma(A,B)-\sigma(X,Y)|<\varepsilon$ whenever $X\subset A$, $Y\subset B$, $|X|\geq\varepsilon|A|$, $|Y|\geq\varepsilon|B|$. Given $\varepsilon>0$, a partition $V(G)=\bigcup_{i=0}^{k}V_i$ is called *ε-regular,* if $|V_0|<\varepsilon n$, $|V_1|=\cdots=|V_k|$, and for every $i\in[k]$, at least $(1-\varepsilon)k$ pairs (V_i,V_j) are ε-regular for $j\in[k]\setminus\{i\}$.

Let $y, x_1, \ldots, x_k$ be real variables. The notation $y \ll (x_1, \ldots, x_k)$ is equivalent to "$y > 0$ and y is sufficiently small, given $x_1, \ldots, x_k$" or, in other words, "there exists a function $y_0(x_1, \ldots, x_k) > 0$ such that, in the subsequent proof, all relations involving $x_1, \ldots, x_k$, and y are true if $y_0(x_1, \ldots, x_k)$ is defined and $0 < y \le y_0(x_1, \ldots, x_k)$".

Likewise, $y \gg (x_1, \ldots, x_k)$ is equivalent to "y is sufficiently large, given $x_1, \ldots, x_k$" or, in other words, "there exists a function $y_0(x_1, \ldots, x_k) > 0$ such that, in the subsequent proof, all relations involving $x_1, \ldots, x_k$, and y are true if $y_0(x_1, \ldots, x_k)$ is defined and $y \ge y_0(x_1, \ldots, x_k)$". Since explicit bounds on $y_0(x_1, \ldots, x_k)$ are often cumbersome and of little use, we believe that the above notation simplifies the presentation and emphasizes the dependence between the relevant variables.

Next we present some known preliminary results we shall need later.

Some preliminary results. The following fact, proved in [22], is a key ingredient of our proof of Theorem 2.1.

Fact 3.1. *For all* $0 < \varepsilon < 1$, $p \ge 2$, *there exist* $\varsigma = \varsigma(\varepsilon, p) > 0$ *and* $L = L(\varepsilon, p)$ *such that for every graph* G *of sufficiently large order* n *with* $k_p(G) < \varsigma n^p$, *there exists a partition* $V(G) = \bigcup_{i=0}^{L} V_i$ *with the following properties:*

- $|V_0| < \varepsilon n$, $|V_1| = \cdots = |V_L|$;
- $\Delta(G[V_i]) < \varepsilon |V_i|$ *for every* $i \in [k]$.

For a general introduction to the Regularity Lemma of Szemerédi [31] the reader is referred to [2] and [18]. We shall use the following specific form implied by Fact 3.1.

Fact 3.2. *For all* $0 < \varepsilon < 1$, $p \ge 2$, *and* $k_0 \ge 2$, *there exist* $\rho = \rho(\varepsilon, p, k_0) > 0$ *and* $K = K(\varepsilon, p, k_0)$ *such that for every graph* G *of sufficiently large order* n *with* $k_{p+1}(G) < \rho n^{p+1}$, *there exists an* ε*-regular partition* $V(G) = \bigcup_{i=0}^{k} V_i$ *with* $k_0 \le k < K$, *and* $\Delta(G[V_i]) < \varepsilon |V_i|$ *for every* $i \in [k]$.

Also we shall use the following simplified versions of the Counting Lemma.

Fact 3.3. *Let* $0 < \varepsilon < d < 1$ *and* (A, B) *be an* ε*-regular pair with* $\sigma(A, B) \ge d$. *Then there are at least* $(1 - \varepsilon)|A|$ *vertices* $v \in A$ *with* $|\Gamma(v) \cap B| \ge d - \varepsilon$.

Fact 3.4. *For all* $0 < d < 1$ *and* $p \ge 2$, *there exist* ε_0 *and* t_0 *such that the following assertion holds:*

Let $\varepsilon > \varepsilon_0$, $t > t_0$, G *be a graph of order* pt, *and* $V(G) = \bigcup_{i=1}^{p} V_i$ *be a partition such that* $|V_1| = \cdots = |V_p| = t$. *If for every* $1 \le i < j \le p$ *the pair* (V_i, V_j) *is* ε*-regular and* $\sigma(V_i, V_j) \ge d$, *then* $k_p(G) \ge d^{p^2} t^p$.

The following lemma can be traced back to Kostochka and Rödl [19]; it was used later by other researchers in various forms. We prove the lemma in Section 3.4.

Lemma 3.5. *For all $k \geq 2$, $d > 0$, $\lambda > 0$ there exists $a = a(k,d,\lambda) > 0$ such that for every graph G and nonempty disjoint sets $U_1, U_2 \subset V(G)$ with $e(G) \geq d|U_1||U_2|$, and sufficiently large $|U_1|$ there exists $W \subset U_1$ with $|W| \geq |U_1|^{1-\lambda}$ and $d(X) > a|U_2|$ for every $X \subset W^{(k)}$.*

3.1. Proof of Theorem 2.1

Set $N = p(n-1)+1$ and let $E(K_N) = E(R) \cup E(B)$ be a 2-coloring. In the following list we show how the variables used in our proof depend on each other

$$\begin{aligned}
\alpha &\ll (p,q),\\
\theta &\ll (p,q,\alpha),\\
\xi &\ll (p,q,\alpha,\gamma),\\
\beta &\ll (p,q,\alpha,\gamma,\xi),\\
\varepsilon &\ll (p,q,\alpha,\beta,\gamma,\xi),\\
k_0 &\gg (p,q,\alpha,\beta,\gamma,\xi,\varepsilon),\\
\eta &\ll (p,q,k_0,\alpha,\beta,\gamma,\xi,\varepsilon),\\
c &\ll (p,q,k_0,\alpha,\beta,\gamma,\xi,\varepsilon),\\
n &\gg (p,q,k_0,\alpha,\beta,\gamma,\xi,\varepsilon).
\end{aligned}$$

Let $K(\varepsilon,p,k_0)$, $\rho(\varepsilon,p,k_0)$, $\varsigma(\varepsilon,p)$, $L(\varepsilon,p)$, $a(k,d,\lambda)$ be as defined in Fact 3.1, Fact 3.2, and Lemma 3.5. Assume that

$$js_{p+1}(R) \leq cn^{p-1} \tag{1}$$

and select a q-degenerate (γ,η)-splittable graph H of order n. To prove the theorem, we shall show that $H \subset B$.

Assumption (1) implies that

$$k_{p+1}(R) \leq \binom{p+1}{2}^{-1}\binom{N}{2} js_{p+1}(R) < cN^{p+1} < \rho(\varepsilon,p+1,k_0)N^{p+1}.$$

Thus, by Fact 3.2, there exists an ε-regular partition $V(R) = \bigcup_{i=0}^{k} V_i$ such that $k_0 < k < K(\varepsilon,p+1,k_0)$ and $\Delta(R[V_i]) < \varepsilon|V_i|$ for all $i \in [k]$. Set $t = |V_1| = \cdots = |V_k|$ and note that for all $i \in [k]$,

$$\delta(B[V_i]) = t - 1 - \Delta(R[V_i]) > t - \varepsilon t - 1 > (1-\beta)t. \tag{2}$$

Next define the graphs R^* and B^* by

$$V(B^*) = [k], \quad V(R^*) = [k],$$
$$E(B^*) = \big\{\{u,v\} : 1 \leq u < v \leq k \text{ and } \sigma_B(V_u, V_v) > 1 - \beta\big\},$$
$$E(R^*) = \big\{\{u,v\} : 1 \leq u < v \leq k,\ (V_u, V_v) \text{ is } \varepsilon\text{-regular and } \sigma_R(V_u, V_v) \geq \beta\big\}.$$

Note first that $E(B^*) \cap E(R^*) = \varnothing$. Moreover, for every vertex $u \in [k]$, we have

$$d_{B^*}(u) + d_{R^*}(u) > k - 1 - \varepsilon k. \tag{3}$$

Indeed, if $\{u,v\} \notin E(B^*) \cup E(R^*)$ then the pair (V_u, V_v) is not ε-regular; hence $\{u,v\} \notin E(B^*) \cup E(R^*)$ holds for fewer than εk vertices $v \in [k] \setminus \{u\}$.

We first show that $H \subset B$ if $\Delta(B^*)$ satisfies

$$\Delta(B^*) \geq (1 + 2\xi)\frac{k}{p}. \tag{4}$$

Indeed, set $r = \Delta(B^*)$ and select $v_0 \in [k]$ with $d_{B^*}(v_0) = r$. Let $\Gamma_{B^*}(v_0) = \{v_1, \ldots, v_r\}$ and set $U_j = V_{v_j}$ for $j = 0, \ldots, r$.

To simplify the presentation of our proof we formulate various claims proved later in KusubsubsectionSection3.1.2.

Claim 3.6. $H \subset B\left[\bigcup_{i=0}^{r} U_i\right]$.

Hereafter we shall assume that (4) fails, i.e.,

$$\Delta(B^*) < (1 + 2\xi)\frac{k}{p}. \tag{5}$$

In view of (3), this inequality implies a lower bound on $\delta(R^*)$, viz.

$$\delta(R^*) > k - 1 - \varepsilon k - (1 + 2\xi)\frac{k}{p} > \left(\frac{p-1}{p} - 2\xi\right)k. \tag{6}$$

In turn, the bound (6), together with the assumption (1), implies a definite structure in R^*.

Claim 3.7. *R^* is p-partite.*

Write $Z_1, \ldots, Z_p$ for the color classes of R^*. For every $i \in [k]$, let $\mu(i) \in [p]$ be the unique value satisfying $i \in Z_{\mu(i)}$. Observe that the sets $Z_1, \ldots, Z_p$ determine a partition of $[N] \setminus V_0$ into p sets that are dense in B. Indeed, $e_B(V_u) > (1 - \beta)t^2/2$ for all $u \in [k]$, and also $e_B(V_u, V_v) > (1 - \beta)t^2$ whenever $\mu(u) = \mu(v)$ and $u \neq v$.

Next we show that the color classes of R^* cannot be two small. Indeed, in view of (5), for every $i \in [p]$, we have

$$|Z_i| = k - \sum_{j\in[p-1]\setminus\{i\}} |Z_j| \geq k - (p-1)(\Delta(B^*)+1) \tag{7}$$
$$= k - (1+2\xi)\frac{(p-1)k}{p} - p + 1 > (1-2p\xi)\frac{k}{p}.$$

In Claims 3.8–3.13 we show that $H \subset B$ provided the inequality

$$\sum_{1\leq h<s\leq p}\left(\sum_{i\in Z_h,\ j\in Z_s} e_B(V_i,V_j)\right) \geq \frac{\alpha}{2}N^2 \tag{8}$$

holds. Inequality (8) implies that $e_B(V_u,V_v)$ is substantial for substantially many pairs u,v with $\mu(u)\neq\mu(v)$; we shall use this fact to embed a substantial part of H. Let us first derive a more specific condition from (8).

Claim 3.8. *There exist $i_1 \in [k]$ and $j \in [p]\setminus\mu(i_1)$ such that*

$$\sum_{\mu(s)=j} e_B(V_{i_1},V_s) > \alpha|Z_j|t^2.$$

We may and shall assume that $i_1 \in Z_1$ and $j=2$. Setting $X = \cup\{V_s : s \in Z_2\}$, we see that Claim 3.8 amounts to

$$e_B(V_{i_1},X) > \alpha|Z_2|t^2. \tag{9}$$

Observe also that, in view of (7), we have

$$|X| = |Z_2|t \geq (1-2p\xi)\frac{kt}{p}. \tag{10}$$

In addition,

$$2e_B(X) = 2\sum_{s\in Z_2} e_B(V_s) + \sum_{i\in Z_2}\left(\sum_{j\in Z_2\setminus\{i\}} e_B(V_i,V_j)\right)$$
$$> |Z_2|(1-\beta)t^2 + |Z_2|(|Z_2|-1)(1-\beta)t^2 = |Z_2|^2(1-\beta)t^2,$$

and so

$$\sigma_B(X) > (1-\beta). \tag{11}$$

Inequality (9) implies that substantially many vertices in V_{i_1} are joined to substantially many vertices in X. In the following claim we strengthen this condition.

Claim 3.9. *There exists* $W_0 \subset V_{i_1}$ *with* $|W_0| > (\alpha/2)t$ *such that for all* $u \in W_0$,

$$|\Gamma_B(u) \cap X| > (\alpha/2)\,|X|.$$

Next set $Y = \cup\{V_s\colon s \in Z_1,\ s \neq i_1\}$; by (7) we have

$$\begin{aligned} |Y| = (|Z_1|-1)t &\geq |Z_1|\left(1-\frac{1}{|Z_1|}\right)t \\ &\geq |Z_1|\left(1-\frac{p}{(1+2\xi)\,k_0}\right)t \geq (1-\beta)\,|Z_1|t. \end{aligned} \tag{12}$$

In addition,

$$\begin{aligned} 2e_B(Y) &= 2\sum_{s\in Z_1\setminus\{i_1\}} e_B(V_s) + \sum_{i\in Z_1\setminus\{i_1\}}\left(\sum_{j\in Z_1\setminus\{i,i_1\}} e_B(V_i,V_j)\right) \\ &> (|Z_1|-1)\,(1-\beta)\,t^2 + (|Z_1|-1)(|Z_1|-2)\,(1-\beta)\,t^2 \\ &= (|Z_1|-1)^2\,(1-\beta)\,t^2, \end{aligned}$$

and so

$$\sigma_B(Y) > (1-\beta). \tag{13}$$

Inequality (12) implies that substantially many vertices in W_0 are joined to substantially many vertices in Y. Next we strengthen this condition.

Claim 3.10. *There exists* $W_1 \subset W_0$ *with* $|W_1| > (\alpha/4)t$ *such that for all* $u \in W_1$,

$$|\Gamma_B(u) \cap Y| > \left(1-\sqrt{\beta}\right)|Y|.$$

Furthermore, the lower bound on $\delta(R^*)$ given by inequality (6) implies that i_1 belongs to a p-clique in R^*.

Claim 3.11. *There exist* $i_2 \in Z_2, \ldots, i_p \in Z_p$ *such that* $\{i_1, i_2, \ldots, i_p\}$ *induces a clique in* R^*.

Claim 3.11, together with $js_{p+1}(R) < cn^{p-1}$, implies that the graph $B[W_1]$ contains a large clique.

Claim 3.12. *There exists* $W \subset W_1$ *with* $|W| \geq t^{1-\gamma/2}$ *such that* $B[W]$ *is a complete graph.*

In summary, Claims 3.8–3.12 together with (10) and (12) imply that the sets W, X, and Y have the following properties:

– $|W| \geq t^{1-\gamma/2}$ and $B[W]$ is a complete graph,
– $|X| \geq (1-2p\xi)k/p$,
– $|Y| \geq (1-2p\xi)k/p$,
– $|\Gamma_B(u) \cap X| > (\alpha/4)|X|$ and $|\Gamma_B(u) \cap Y| > (1-\sqrt{\beta})|Y|$, for every $u \in W$.

It turns out that these properties are sufficient to achieve our goal – to embed H.

Claim 3.13. *$H \subset B[W \cup X \cup Y]$.*

Hereafter we shall assume that (8) fails, i.e.,

$$\sum_{1 \leq h < s \leq p} \left(\sum_{i \in Z_h,\, j \in Z_s} e_B(V_i, V_j) \right) < \frac{\alpha}{2} N^2. \tag{14}$$

This inequality implies that $e_B(V_u, V_v)$ is small for most pairs u, v with $\mu(u) \neq \mu(v)$. We shall deduce that R can be made p-partite by removing only a small proportion of its vertices.

Claim 3.14. *R contains an induced p-partite subgraph R_1 with color classes $U_1, \ldots, U_p$ such that $|U_1| = \cdots = |U_p| > (1-\theta)n$ and*

$$|\Gamma_{R_1}(u) \cap U_i| > (1-\theta)\, n$$

for each $i \in [p]$ and $u \in V(R_1) \setminus U_i$.

Since R_1 is an induced p-partite subgraph of R, the graph B contains cliques of size close to n; hence H can be embedded in B almost entirely; to embed H in full, we need an additional argument. Analyzing the way vertices from $V(R) \setminus V(R_1)$ can be joined to the vertices of R_1, we derive the following assertion.

Claim 3.15. *There exist disjoint sets $M \subset V(R_1)$ and $A, C \subset V(R) \setminus V(R_1)$ such that*

$$|M| + |A| + |C| = n - 1 + \left\lceil \frac{|C|+1}{p} \right\rceil, \tag{15}$$

$$|A| < \theta n, \tag{16}$$

$$|C| < 2\theta n \tag{17}$$

with the following properties:
(i) $B[M]$ is a complete graph;
(ii) $\Gamma_B(u) \cap M = M$ for every vertex $u \in A$;
(iii) $|\Gamma_B(u) \cap M| \geq (1-p^2\theta)|M|$ for every vertex $u \in C$.

Using the properties of the sets M, A, and C we embed H, completing the proof of the theorem.

Claim 3.16. *$H \subset B[M \cup A \cup C]$.*

3.1.1. Results supporting proofs of the claims

Fact 3.17. *Every subgraph of a q-degenerate graph is q-degenerate.*

Fact 3.18. *The vertices of any q-degenerate graph of order n can be labeled* $\{v_1,\dots,v_n\}$ *so that* $|\Gamma(v_i)\cap\{v_1,\dots,v_{i-1}\}|\le q$ *for every* $i\in[n]$.

Fact 3.19. *Every q-degenerate graph is* $(q+1)$*-partite.*

Proposition 3.20. *In any q-degenerate graph H the number of vertices of degree* $2q+1$ *or higher is at most* $2q|H|/(2q+1)$.

Proof. Letting $S=\{u\colon u\in V(H),\ d(u)\ge 2q+1\}$, we have

$$2q|H| \ge 2e(H) \ge \sum_{u\in V(H)} d(u) \ge \sum_{u\in S} d(u) \ge (2q+1)|S|,$$

and the assertion follows. ∎

Lemma 3.21. *Let* $q\ge 0$, $\tau>0$, *and* $G=G(n)$ *be a graph with* $\delta(G)\ge(1-\tau)n$. *Then G contains all q-degenerate graphs of order* $l\le(1-q\tau)n$.

Proof. We use induction on l. The assertion holds trivially for $l=1$; assume that it holds for $1\le l'<l$. Let H be a q-degenerate graph of order l and $u\in V(H)$ be a vertex with $d_H(u)=d\le q$. Let $\Gamma_H(u)=\{v_1,\dots,v_d\}$ and $H'=H-u$. By the induction assumption there exists a monomorphism $\varphi\colon H'\to G$. We have

$$\left|\bigcap_{i=1}^{d}\Gamma_G(\varphi(v_i))\right| \ge \sum_{i=1}^{d} d_G(\varphi(v_i)) - (d-1)n > d(1-\tau)n-(d-1)n$$
$$\ge (1-q\tau)n > l'.$$

Hence there exists $v\in\left(\bigcap_{i=1}^{d}\Gamma_G(\varphi(v_i))\right)\setminus\varphi(V(H'))$. To complete the induction step and the proof, define a monomorphism $\varphi'\colon H\to G$ by

$$\varphi'(w)=\begin{cases}\varphi(w), & \text{if } w\in V(H')\\ v, & \text{if } w=u.\end{cases}$$

∎

Lemma 3.22. *Suppose G is a* (γ,η)*-splittable q-degenerate graph of order n. Then there exists* $M\subset V(G)$ *such that* $|M|<(2q+1)n^{1-\gamma}$, *and* $\psi(G-M)<\eta n$ *and* $|\Gamma(u)\cap M|\le 2q$ *for every* $u\in V(G)\setminus M$.

Proof. Since G is (γ,η)-splittable, there is a set $N\subset V(G)$ such that $|N|<n^{1-\gamma}$ and $\psi(G-N)<\eta n$. Set $M=N$ and apply the following procedure to G:

While there exists $u\in V(G)\setminus M$ *with* $|\Gamma(u)\cap M|\geq 2q+1$ *do*
$M:=M\cup\{u\}$
end.

Set $M'=\{u\colon u\in M,\ |\Gamma(u)\cap M|\geq 2q+1\}$. Since $G[M]$ is q-degenerate, Proposition 3.20 implies that $|M'|\leq 2q|M|/(2q+1)$. By our selection, $|\Gamma(u)\cap M|\geq 2q+1$ for all of $u\in M\setminus N$; hence, $|M\setminus N|\leq 2q|M|/(2q+1)$, implying that $|M|\leq(2q+1)|N|\leq(2q+1)n^{1-\gamma}$. ∎

Proposition 3.23. *Let* $0<\tau<1$ *and* G *be a graph of order* n *with* $e(G)>(1-\tau)n^2/2$. *Then* G *contains an induced subgraph* G_0 *with* $|G_0|>(1-\sqrt{\tau})n$ *and* $\delta(G_0)>(1-2\sqrt{\tau})n$.

Proof. Let
$$W=\left\{u\colon d_G(u)>\left(1-\sqrt{\tau}\right)n\right\}.$$
We have
$$\begin{aligned}(1-\tau)\,n^2<2e(G)&=\sum_{u\in W}d_G(u)+\sum_{u\in V(G)\setminus W}d_G(u)\\&\leq n|W|+\left(1-\sqrt{\tau}\right)n(n-|W|)\\&=\sqrt{\tau}n|W|+\left(1-\sqrt{\tau}\right)n^2,\end{aligned}$$
and so $(1-\sqrt{\tau})n<|W|$. Furthermore, for every $u\in W$,
$$|\Gamma_G(u)\cap W|\geq|\Gamma_G(u)|-|V(G)\setminus W|\geq\left(1-2\sqrt{\tau}\right)n.$$
Thus, setting $G_0=G[W]$, the proof is completed. ∎

Fact 3.24 **([3]).** *Let* $p\geq 3$, $n>p^8$, *and* $0<\alpha<p^{-8}/16$. *If a graph* $G=G(n)$ *satisfies*
$$e(G)>\left(\frac{p-1}{2p}-\alpha\right)n^2,$$
then either
$$js_p(G)>\left(1-\frac{1}{p^3}\right)\frac{n^{p-2}}{p^{p+5}},\tag{18}$$
or G *contains an induced* p*-partite subgraph* G_0 *of order at least* $(1-2\sqrt{\alpha})n$ *with minimum degree*
$$\delta(G_0)>\left(1-\frac{1}{p}-4\sqrt{\alpha}\right)n.\tag{19}$$

Fact 3.25 ([3]). *Let* $2 \le r < \omega(G)$ *and* $\alpha \ge 0$. *If* $G = G(n)$ *and*

$$\delta(G) \ge \Big(\frac{r-1}{r} + \alpha\Big)n$$

then

$$k_{r+1}(G) \ge \alpha \frac{r^2}{r+1}\left(\frac{n}{r}\right)^{r+1}.$$

3.1.2. Proofs of the claims

Let $K(\varepsilon, p, k)$ and $\rho(\varepsilon, p, k)$, $\varsigma(\varepsilon, p)$, and $L(\varepsilon, p)$, be as defined in Fact 3.1 and Fact 3.2; set $K = K(\varepsilon, p, k_0)$.

Proof of Claim 3.6. Set $\varsigma = \varsigma(1/(2q), p)$ and $L = L(1/(2q), p)$.

Note first that the sets $U_0, \dots, U_r$ satisfy the following conditions:

- $|U_0| = \dots = |U_r| = t$;
- $\delta(B[U_i]) \ge (1-\varepsilon)t > (1-\beta)t$ for $i = 0, \dots, r$;
- $\sigma_B(U_0, U_i) > 1 - \beta$ for $i = 1, \dots, r$.

For every $u \in U_0$ set

$$D(u) = \big\{i\colon i \in [r],\ \ |\Gamma(u) \cap U_i| \ge \big(1 - 2\sqrt{\beta}\,\big)t\big\}$$

and let

$$W = \big\{u\colon u \in U_0,\ |D(u)| \ge \big(1 - \sqrt{\beta}\,\big)r\big\}.$$

We shall prove that $|W| > t/2$. Indeed, we see that

$$\begin{aligned}
(1-\beta)\, rt^2 < \sum_{i \in [r]} e(U_0, U_i) &= \sum_{u \in U_0} \left(\sum_{i \in [r]} |\Gamma(u) \cap U_i| \right) \\
&= \sum_{u \in W} \left(\sum_{i \in [r]} |\Gamma(u) \cap U_i| \right) + \sum_{u \in U_0 \setminus W} \left(\sum_{i \in [r]} |\Gamma(u) \cap U_i| \right) \\
&< |W| rt + \sum_{u \in U_0 \setminus W} \big(D(u)t + (r - D(u))\big(1 - 2\sqrt{\beta}\,\big)t\big) \\
&\le |W| rt + t(t - |W|)\big(r\big(1 - 2\sqrt{\beta}\,\big) + 2\sqrt{\beta} D(u)\big) \\
&< |W| rt + tr(t - |W|)\big(\big(1 - 2\sqrt{\beta}\,\big) + 2\sqrt{\beta}\big(1 - \sqrt{\beta}\,\big)\big) \\
&= |W| rt + rt(t - |W|)(1 - 2\beta).
\end{aligned}$$

Hence

$$(1-\beta)\, t \le |W| + (t - |W|)\,(1 - 2\beta) = t\,(1 - 2\beta) + 2\beta|W|,$$

and so $|W| > t/2$.

Since $D(u) \subset [r]$, the pigeonhole principle gives $D \subset [r]$ and $X \subset W$ such that

$$|X| \geq |W|/2^r \geq t/2^{K+1}$$

and $D(u) = D$ for every $u \in X$. Since

$$js_{p+1}(R[X]) < cn^{p-1} \leq c\left(\frac{N}{p}\right)^{p-1} \leq c\left(\frac{Kt}{p(1-\varepsilon)}\right)^{p-1} \leq c(Kt)^{p-1}$$
$$< c\left(K2^{K+1}|X|\right)^{p-1} \leq \varsigma |X|^{p-1},$$

Theorem 3.1 implies that X contains a set Y with $|Y| \geq |X|/2L$ and

$$\delta(B[Y]) > (1 - 1/2q)|Y|. \tag{20}$$

On the other hand, Lemma 3.22 implies that there exists $M \subset V(H)$ with $|M| \leq (2q+1)|H|^{1-\gamma}$ such that $\psi(H-M) \leq \gamma|H|$ and $|\varGamma_H(u) \cap M| \leq 2q$ for every $u \in V(H-M)$. Since the graph $H[M]$ is q-degenerate, we have

$$|M| \leq (2q+1)|H|^{1-\gamma} \leq (2q+1)(rt)^{1-\gamma} \leq \frac{t}{2^{r+3}L} \leq \frac{|X|}{4L} \leq \frac{|Y|}{2}$$

for t large. Hence, in view of (20), Lemma 3.21 implies that there exists a monomorphism $\varphi\colon H[M] \to Y$. We shall extend φ to H by mapping each component of $H-M$ in turn.

Select a component C of $H-M$. The choice of the set M implies that

$$|C| \leq \psi(H-M) \leq \eta|H| < \frac{\sqrt{\beta}|H|}{K} < \frac{\sqrt{\beta}rt}{r} = \sqrt{\beta}t.$$

We shall extend φ over C by mapping C in any set U_i, $i \in D$ in which there are at least $(6q+1)\sqrt{\beta}t$ vertices outside of the current range of φ. Set $l = |C|$; Proposition 3.18 implies that the vertices of C can be arranged as $v_1, \ldots, v_l$ so that $|\varGamma_H(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq q$ for every $i \in [l]$. We shall extend φ over C mapping each $v_i \in V(C)$ in turn. Suppose we have mapped $v_1, \ldots, v_{i-1}$. The vertex v_i is joined to at most q vertices from $\{v_1, \ldots, v_{i-1}\}$ and at most $2q$ vertices from M, i.e.,

$$v_i \in \left(\bigcap_{j=1}^{h} \varGamma_H\left(v_{i_j}\right)\right) \cap \left(\bigcap_{j=1}^{s} \varGamma_H\left(u_{i_j}\right)\right),$$

where $v_{i_1},\ldots,v_{i_h} \in \{v_1,\ldots,v_{i-1}\}$, $h \le q$, and $u_{i_1},\ldots,u_{i_s} \in M$, $s \le 2q$. Set for convenience $x_j = \varphi(v_{i_j})$ for all $j \in [h]$, and $y_j = \varphi(u_{i_j})$ for all $j \in [s]$. Note that

$$\begin{aligned}
&\left(\bigcap_{j=1}^{h} \Gamma_B(x_j) \cap U_i\right) \cap \left(\bigcap_{j=1}^{s} \Gamma_B(y_j) \cap U_i\right) \\
&\ge \sum_{j\in[h]} |\Gamma_B(x_j) \cap U_i| + \sum_{j\in[s]} |\Gamma_B(y_j) \cap U_i| - (h+s-1)t \\
&> (h+s)\big(1-2\sqrt{\beta}\big)t - (h+s-1)t \\
&> \big(1-6q\sqrt{\beta}\big)t > \big(1-(6q+1)\sqrt{\beta}\big)t + |C|.
\end{aligned}$$

Hence there is a vertex $z \in U_i$ that is joined to the vertices $x_1,\ldots,x_h,y_1,\ldots,y_s$ and is outside the current range of φ. Setting $\varphi(v_i) = z$, we extend φ to a monomorphism that maps v_i into B as well. In this way φ can be extended over the whole component C.

Assume for a contradiction that φ cannot be extended over some component C. Therefore, for every $i \in D$, the current range of φ contains at least $(1-(6q+1)\sqrt{\beta})t$ vertices from U_i. Hence

$$\begin{aligned}
|H| &\ge |D|\big(1-(6q+1)\sqrt{\beta}\big)t > \big(1-\sqrt{\beta}\big)\big(1-(6q+1)\sqrt{\beta}\big)rt \\
&\ge \big(1-(6q+2)\sqrt{\beta}\big)rt > (1-\xi)rt \ge \frac{(1-\xi)(1+2\xi)}{p}kt \\
&\ge (1-\xi)\,(1+2\xi)\,(1-\varepsilon)\,n > n,
\end{aligned}$$

a contradiction, completing the proof. ∎

Proof of Claim 3.7. Let $\upsilon = \beta^{p^2}$.

We shall prove first that $\omega(R^*) \le p$. Otherwise by Lemma 3.4 we have

$$k_{p+1}(R) \ge \upsilon t^{p+1} \ge \upsilon\,(1-\varepsilon)^{p+1}\left(\frac{N}{K}\right)^{p+1},$$

and so

$$\begin{aligned}
js_{p+1}(R) &\ge \frac{\binom{p+1}{2}}{\binom{N}{2}} k_{p+1}(R) > \upsilon\frac{1}{N^2}\,(1-\varepsilon)^{p+1}\left(\frac{N}{K}\right)^{p+1} \\
&\ge \upsilon\frac{(1-\varepsilon)^{p+1}}{K^{p+1}} N^{p-1} > cn^{p-1},
\end{aligned}$$

contradicting (1).

Since $\omega(R^*)\leq p$, and

$$\delta\left(R^*\right) > \left(1-\frac{1}{p}-2\xi\right)k \geq \left(1-\frac{1}{p-1/3}\right)k,$$

by a well-known theorem of Andrásfai, Erdős, and Sós [1], R^* is p-partite. ∎

Proof of Claim 3.8. In view of (8), we have

$$\begin{aligned}\sum_{h\in[p]}\left(\sum_{i\in Z_h,\, j\in[k]\setminus Z_h} e_B(V_i,V_j)\right) &= \sum_{h\in[p]}\left(\sum_{s\in[p]\setminus\{h\}}\left(\sum_{i\in Z_h,\, j\in Z_s} e_B(V_i,V_j)\right)\right)\\ &= 2\sum_{1\leq h<s\leq p}\left(\sum_{i\in Z_h,\, j\in Z_s} e_B(V_i,V_j)\right)\geq \alpha N^2.\end{aligned}$$

Hence, we can select $h\in[p]$ so that

$$\sum_{i\in Z_h}\left\{e_B(V_i,V_j)\colon j\in[k]\setminus Z_h\right\}\geq\frac{\alpha N^2}{p}.$$

Since by (5) we have

$$|Z_h|\leq \Delta\left(B^*\right)+1 < (1+2\xi)\,\frac{k}{p}+1\leq(1+3\xi)\,\frac{k}{p},$$

there is an $i_1\in Z_h$ such that

$$\sum_{j\in[k]\setminus Z_h} e_B\left(V_{i_1},V_j\right)\geq\frac{\alpha N^2}{(1+3\xi)\,k}\geq\frac{\alpha N}{(1+3\xi)}t,$$

and so,

$$\sum_{j\in[p]\setminus\{h\}}\left(\sum_{\mu(s)=j} e_B\left(V_{i_1},V_s\right)\right)\geq\frac{\alpha N}{(1+3\xi)}t.$$

Furthermore, in view of (7) we have

$$\sum_{j\in[p]\setminus\{h\}}|Z_j| = k-|Z_h|\leq k-(1-2p\xi)\,\frac{k}{p}=\frac{p-1+2p\xi}{p}k,$$

and thus

$$\frac{N}{(1+3\xi)} > \left(\frac{p-1+2p\xi}{p}\right)N\geq\left(\frac{p-1+2p\xi}{p}\right)kt\geq t\sum_{j\in[p]\setminus\{h\}}|Z_j|.$$

Therefore,

$$\sum_{j\in[p]\setminus\{h\}}\left(\sum_{\mu(s)=j} e_B\left(V_{i_1},V_s\right)\right)\geq \alpha t^2 \sum_{j\in[p]\setminus\{h\}}|Z_j|$$

and the pigeonhole principle gives some $j\in[p]\setminus\{h\}$ for which

$$\sum_{\mu(s)=j} e_B\left(V_{i_1},V_s\right)>\alpha|Z_j|t^2,$$

completing the proof. ∎

Proof of Claim 3.9. Set

$$W_0=\left\{u\colon u\in V_{i_1},\ |\Gamma_B(u)\cap X|>\frac{\alpha}{2}|X|\right\}.$$

In view of (9),

$$\alpha|X|t<\sum_{u\in V_{i_1}}|\Gamma_B(u)\cap X|=\sum_{u\in W_0}|\Gamma_B(u)\cap X|+\sum_{u\in V_{i_1}\setminus W_0}|\Gamma_B(u)\cap X|$$
$$<|W_0||X|+(t-|W_0|)\frac{\alpha}{2}|X|,$$

implying that

$$\frac{\alpha}{2}t<\left(1-\frac{\alpha}{2}\right)|W_0|,$$

so $|W_0|>(\alpha/2)t$. ∎

Proof of Claim 3.10. Let

$$W=\left\{u\colon u\in V_{i_1},\ |\Gamma_B(u)\cap Y|>\left(1-\sqrt{\beta}\right)|Y|\right\}.$$

We shall show that $|W|>(1-\sqrt{\beta})t$. Indeed,

$$(1-\beta)|Y|t<\sum_{s\in Z_1\setminus\{i_1\}} e\left(V_{i_1},V_s\right)=e\left(V_{i_1},Y\right)=\sum_{u\in V_{i_1}}|\Gamma_B(u)\cap Y|$$
$$=\sum_{u\in W}|\Gamma_B(u)\cap Y|+\sum_{u\in V_{i_1}\setminus W}|\Gamma_B(u)\cap Y|$$
$$<|W||Y|+(t-|W|)\left(1-\sqrt{\beta}\right)|Y|.$$

Hence

$$(1-\beta)\,t<|W|+(t-|W|)\left(1-\sqrt{\beta}\right),$$

so $|W|>(1-\sqrt{\beta})t$.

Now $W_1 = W_0 \cap W$ satisfies

$$|W_1| \geq |W_0| + |W| - t > \left(\frac{\alpha}{2} - \sqrt{\beta}\right) t \geq \frac{\alpha}{4} t,$$

completing the proof. ∎

Proof of Claim 3.11. Let $\{i_1, \ldots, i_s\}$ induces a maximal clique in R^* containing i_1; assume for a contradiction that $s < p$. Then by (6),

$$d_{R^*}(\{i_1, \ldots, i_s\}) \geq \sum_{j=1}^{s} d_{R^*}(i_j) - (s-1)k > s\left(\frac{p-1}{p} - 2\xi\right) k - (s-1)k > 0.$$

Thus, there is a vertex $i \in [k]$ joined in R^* to all vertices $i_1, \ldots, i_s$, contradicting the fact that $\{i_1, \ldots, i_s\}$ induces a maximal clique and completing the proof. ∎

Proof of Claim 3.12. For $s = 2, \ldots, p$, applying Lemma 3.3, select $P_s \subset V_{i_1}$ with $|P_s| \geq (1-\varepsilon)t$ and $|\Gamma_R(u) \cap V_{i_s}| > (\beta - \varepsilon)t$ for every $u \in P_s$. Hence

$$\left|\bigcap_{s=2}^{p} P_s\right| > (p-1)(1-\varepsilon)t - (p-2)t > (1 - p\varepsilon)t.$$

Therefore, for $W_2 = W_1 \cap (\bigcap_{s=2}^{p} P_s)$ we have

$$|W_2| = \left|W_1 \cap \left(\bigcap_{s=2}^{p} P_s\right)\right| \geq |W_1| + \left|\bigcap_{s=2}^{p} P_s\right| - t$$
$$\geq (\alpha/4)t + (1 - p\varepsilon)t - t \geq (\alpha/8)t.$$

Set $Q_1 = W_2$ and let $a = a(2, \beta/2, \gamma/(2p))$ (see Lemma 3.5).

For $s = 2, \ldots, p$, applying Lemma 3.5 with $k = 2$, $d = \beta/2$, $\lambda = \gamma/(2p)$, find $Q_s \subset Q_{s-1}$ with $|Q_s| \geq |Q_{s-1}|^{1-\gamma/(2p)}$ and $|\Gamma_R(uv) \cap V_{i_s}| > at$ for every $\{u, v\} \in Q_s^{(2)}$. Set $W = Q_p$ and note that

$$|W| = |Q_p| \geq |Q_{p-1}|^{1-\gamma/(2p-2)} \geq \cdots \geq |Q_1|^{(1-\gamma/2p)^{p-1}} \geq |Q_1|^{1-\gamma(p-1)/(2p)}$$
$$\geq \left(\frac{\alpha}{8}t\right)^{1-\gamma(p-1)/(2p)} > t^{1-\gamma/2},$$

for t sufficiently large.

Assume for a contradiction that $R[W]$ contains an edge uv. Since $|\Gamma_R(uv)\cap V_{i_s}|>at$, by Lemma 3.4 we have

$$js_{p+1}(R) \geq ((a-\varepsilon)\,t)^{p-1} > \left((a-\varepsilon)\,\frac{N\,(1-\varepsilon)}{K}\right)^{p-1} > \left(\frac{(a-\varepsilon)\,(1-\varepsilon)}{K}\right)^{p-1} N^{p-1} > cn^{p-1},$$

a contradiction with (1). So W is a clique in B, completing the proof. ∎

Proof of Claim 3.13. Since (11) and (13) imply that $e_B(X)>(1-\beta)|X|^2/2$ and $e_B(Y)>(1-\beta)|Y|^2/2$, by Proposition 3.23, there exist $X_0\subset X$ and $Y_0\subset Y$ such that

$$|X_0| > \left(1-\sqrt{\beta}\,\right)|X| > \left(1-\sqrt{\beta}\,\right)(1-2p\xi)\frac{k}{p}t \geq (1-3p\xi)\frac{k}{p}t,$$
$$\delta(B[X_0]) > \left(1-2\sqrt{\beta}\,\right)|X_0|,$$
$$|Y_0| > \left(1-\sqrt{\beta}\,\right)|Y| > \left(1-\sqrt{\beta}\,\right)(1-2p\xi)\frac{k}{p}t \geq (1-3p\xi)\frac{k}{p}t,$$
$$\delta(B[Y_0]) > \left(1-2\sqrt{\beta}\,\right)|Y_0|.$$

Also, for every $u\in W$,

$$|\Gamma_B(u)\cap X_0| \geq |\Gamma_B(u)\cap X| - |X\setminus X_0| \geq \frac{\alpha}{4}|X| - \sqrt{\beta}|X| > \frac{\alpha}{8}|X_0|,$$
$$|\Gamma_B(u)\cap Y_0| \geq |\Gamma_B(u)\cap Y| - |Y\setminus Y_0| \geq \left(1-\sqrt{\beta}\,\right)|Y| - \sqrt{\beta}|Y| > \left(1-2\sqrt{\beta}\,\right)|Y_0|.$$

Next, Lemma 3.5 implies that there exists $a>0$ and $U\subset W$ such that for every $Q\subset U^{(2q)}$, $|\Gamma_B(Q)\cap X_0|>a|X_0|$ and $|U|>|W|^{1-\gamma/2}$.

Also Lemma 3.22 implies that there exists $M\subset V(H)$ with $|M|\leq(2q+1)|H|^{1-\gamma}$ such that $\psi(H-M)\leq\eta|H|$ and $d_M(u)\leq 2q$ for every $u\in V(H-M)$. Since the graph $H[M]$ is q-degenerate, for t large, we have

$$|M| \leq (2q+1)|H|^{1-\gamma} < (2q+1)(kt)^{1-\gamma} < t^{(1-\gamma/2)^2} < |U|.$$

Let $\varphi\colon H[M]\to U$ be a one-to-one mapping; since $B[U]$ is complete, φ is a monomorphism. We shall extend φ to H by mapping almost all components of $H-M$ into Y_0 and the remaining components into X_0. We can partition $H-M$ into two disjoint graphs H_1 and H_2 such that

$$|H_1| < \left(1-6q\sqrt{\beta}-3p\xi\right)\frac{k}{p}t, \tag{21}$$

$$|H_2| < \left(a-2q\sqrt{\beta}-3p\xi\right)\frac{k}{p}t. \tag{22}$$

Indeed, collect into H_1 as many components of $H-M$ as possible so that (21) still holds, and collect the remaining components into H_2. Since $\psi(H-M) < \eta n$, inequality (22) follows from

$$\begin{aligned}
|H_2| &\leq n - |H_1| \leq n - \left(1 - 6q\sqrt{\beta} - 3p\xi\right)\frac{k}{p}t + \eta n \\
&< (1+2\eta)\,\frac{N}{p} - \left(1 - 6q\sqrt{\beta} - 3p\xi\right)\frac{k}{p}t \\
&< (1+2\eta)\,\frac{(1+2\varepsilon)\,k}{p}t - \left(1 - 6q\sqrt{\beta} - 3p\xi\right)\frac{k}{p}t \\
&< \left(3\eta + 2\varepsilon + 6q\sqrt{\beta} + 3p\xi\right)\frac{k}{p}t < \left(a - 2q\sqrt{\beta} - 3p\xi\right)\frac{k}{p}t.
\end{aligned}$$

Set $l = |H_1|$; Proposition 3.18 implies that the vertices of H_1 can be arranged as $v_1, \ldots, v_l$ so that $|\Gamma_H(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq q$ for every $i \in [l]$. We shall extend φ over H_1 by mapping each $v_i \in V(H_1)$ in turn. Let $\Gamma_H(v_i) = \{v_{i_1}, \ldots, v_{i_h}\} \cup \{u_{i_1}, \ldots, u_{i_s}\}$, where $v_{i_1}, \ldots, v_{i_h} \in \{v_1, \ldots, v_{i-1}\}$, $h \leq q$, and $u_{i_1}, \ldots, u_{i_s} \in M$, $s \leq 2q$. Therefore,

$$v_i \in \left(\bigcap_{j=1}^{h} \Gamma_H\left(v_{i_j}\right)\right) \cap \left(\bigcap_{j=1}^{s} \Gamma_H\left(u_{i_j}\right)\right).$$

Set for convenience $x_j = \varphi(v_{i_j})$ for all $j \in [h]$, and $y_j = \varphi(u_{i_j})$ for all $j \in [s]$. Note that

$$\begin{aligned}
&\left(\bigcap_{j=1}^{h} \Gamma_B(x_j) \cap Y_0\right) \cap \left(\bigcap_{j=1}^{s} \Gamma_B(y_j) \cap Y_0\right) \\
&\quad \geq \sum_{j \in [h]} |\Gamma_B(x_j) \cap Y_0| + \sum_{j \in [s]} |\Gamma_B(y_j) \cap Y_0| - (h+s-1)|Y_0| \\
&\quad > (h+s)\left(1 - 2\sqrt{\beta}\right)|Y_0| - (h+s-1)|Y_0| > \left(1 - 6q\sqrt{\beta}\right)|Y_0| \\
&\quad > \left(1 - 6q\sqrt{\beta}\right)(1 - 3p\xi)\frac{k}{p} > |H_1|.
\end{aligned}$$

Hence, there is a vertex $z \in Y_0$ that is joined to the vertices $x_1, \ldots, x_h$, $y_1, \ldots, y_s$ and is outside the current range of φ. Setting $\varphi(v_i) = z$, we extend φ to a monomorphism that maps v_i into Y_0 as well. In this way φ can be extended over the entire H_1.

Set now $l = |H_2|$; Proposition 3.18 implies that the vertices of H_2 can be arranged as $v_1, \ldots, v_l$ so that $|\Gamma_H(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq q$ for every $i \in [l]$. We shall extend φ over H_2 mapping each $v_i \in V(H_2)$ in turn. Let

$\Gamma_H(v_i) = \{v_{i_1}, \ldots, v_{i_h}\} \cup \{u_{i_1}, \ldots, u_{i_s}\}$ where $v_{i_1}, \ldots, v_{i_h} \in \{v_1, \ldots, v_{i-1}\}$, $h \le q$, and $u_{i_1}, \ldots, u_{i_s} \in M$, $s \le 2q$. Therefore,

$$v_i \in \left(\bigcap_{j=1}^{h} \Gamma_H\left(v_{i_j}\right) \right) \cap \left(\bigcap_{j=1}^{s} \Gamma_H\left(u_{i_j}\right) \right).$$

Set for convenience $x_j = \varphi(v_{i_j})$ for all $j \in [h]$, and $y_j = \varphi(u_{i_j})$ for all $j \in [s]$. Note that

$$\begin{aligned} &\left(\bigcap_{j=1}^{h} \Gamma_B(x_j) \cap X_0 \right) \cap \left(\bigcap_{j=1}^{s} \Gamma_B(y_j) \cap X_0 \right) \\ &\quad \ge a|X_0| + \sum_{j\in[s]} |\Gamma_B(y_j) \cap X_0| - s|X_0| \\ &\quad > a|X_0| + s\left(1 - 2\sqrt{\beta}\right)|X_0| - s|X_0| \\ &\quad > \left(a - 2q\sqrt{\beta}\right)|X_0| > \left(a - 2q\sqrt{\beta}\right)(1 - 3p\xi)\frac{k}{p} > |H_2|. \end{aligned}$$

Hence, there is a vertex $z \in X_0$ that is joined to the vertices $x_1, \ldots, x_h$, $y_1, \ldots, y_s$ and is outside the current range of φ. Setting $\varphi(v_i) = z$, we extend φ to a monomorphism that maps v_i into X_0 as well. In this way φ can be extended over the entire H_2. ∎

Proof of Claim 3.14. In view of (14) and (7),

$$\begin{aligned} e(R) &\ge \sum_{1\le h<s\le p} \left(\sum_{i\in Z_h,\ j\in Z_s} e_R(V_i, V_j) \right) \\ &\ge \sum_{1\le h<s\le p} |Z_h||Z_s|t^2 - \sum_{1\le h<s\le p} \left(\sum_{i\in Z_h,\ j\in Z_s} e_B(V_i, V_j) \right) \\ &\ge \binom{p}{2} (1 - 2p\xi)^2 \frac{k^2t^2}{p^2} - \frac{\alpha}{2}N^2 = \frac{p-1}{2p} (1 - 4p\xi)(1 - \varepsilon)^2 N^2 - \frac{\alpha}{2}N^2 \\ &\ge \left(\frac{p-1}{2p} - 4p\xi - 2\varepsilon - \frac{\alpha}{2} \right) N^2 \ge \left(\frac{p-1}{2p} - \alpha \right) N^2. \end{aligned}$$

On the other hand we have

$$js_{p+1}(R) < cn^{p-1} < \left(1 - \frac{1}{p^3}\right) \frac{N^{p-1}}{p^{p+5}};$$

hence, Fact 3.24 implies that R has a p-partite induced subgraph R_0 with $|R_0|>(1-2\sqrt{\alpha})N$ and

$$\delta(R_0) > \left(1 - \frac{1}{p} - 4\sqrt{\alpha}\right) N. \tag{23}$$

We shall find R_1 as an induced subgraph of R_0. Observe that by (23) every color class of R_0 has at most $N-\delta(R_0) > (1/p+4\sqrt{\alpha})N$ vertices. Hence, every color class of G_0 has at least

$$\left(1-2\sqrt{\alpha}\right) N - (p-1)\left(\frac{1}{p}+4\sqrt{\alpha}\right) N > \left(1-4p(p-1)\sqrt{\alpha}\right)\frac{N}{p} > (1-\theta)\, n$$

vertices. From each color class select a set of $\lceil (1-\theta)n \rceil$ vertices and write R_1 for the graph induced by their union.

Let $u \in V(R_1)$ and U be a color class of R_1 such that $u \notin U$. Since $\delta(R_1) \geq \delta(R_0) - |R_0| + |R_1|$, we see that

$$\begin{aligned}
|\Gamma_{R_1}(u) \cap U| &> |U| + \delta(R_1) - \frac{p-1}{p}|R_1| \\
&\geq |U| + \delta(R_0) - |R_0| + |R_1| - \frac{p-1}{p}|R_1| \\
&= \delta(R_0) - |R_0| + \frac{2}{p}|R_1| \\
&> \left(1 - \frac{1}{p} - 4\sqrt{\alpha}\right) N - N + \left(\frac{2}{p} - 8p\sqrt{\alpha}\right) N \\
&> \left(1 - 8p(p+1)\sqrt{\alpha}\right)\frac{N}{p} \geq (1-\theta)\, n,
\end{aligned}$$

completing the proof. ∎

Proof of Claim 3.15. Set $s=|U_1|=\cdots=|U_p|$. According to Claim 3.14,

$$\begin{gathered}
(1-\theta)\, n < s < n, \\
|\Gamma_R(u) \cap U_i| > (1-\theta)\, n
\end{gathered} \tag{24}$$

for every U_i and every $u \in V(R_1) \setminus U_i$.

Set $X = V(R) \setminus V(R_1)$ and define a partition $X = Y \cup Z$ as follows:

$$\begin{aligned}
&Y = \left\{u \colon u \in X,\ \Gamma_R(u) \cap U_i \neq \varnothing \text{ for every } i \in [p]\right\}, \\
&Z = X \setminus Y.
\end{aligned}$$

We first show that for every $u\in Y$, there exists two distinct color classes U_i and U_j such that

$$|\mathit{\Gamma}_R(u)\cap U_i| \le p^2\theta n, \qquad |\mathit{\Gamma}_R(u)\cap U_j| \le p^2\theta n. \tag{25}$$

For a contradiction, assume the opposite: let $u\in Y$ be such that $|\mathit{\Gamma}_R(u)\cap U_i| > \theta n$ for at least $p-1$ values $i\in[p]$, say for $i=2,\dots,p$. The definition of Y implies that there exists some $v\in U_1\cap\mathit{\Gamma}_R(u)$. Observe that for every $i\in[2..p]$,

$$\begin{aligned}|\mathit{\Gamma}_R(u)\cap\mathit{\Gamma}_R(v)\cap U_i| &\ge |\mathit{\Gamma}_R(u)\cap U_i| + |\mathit{\Gamma}_R(v)\cap U_i| - |U_i|\\ &> p^2\theta n + n - \theta n - s > (p^2-1)\theta n.\end{aligned}$$

Therefore, for every $i\in[2..p]$, we can select a set $W_i\subset\mathit{\Gamma}_R(u)\cap\mathit{\Gamma}_R(v)\cap U_i$ with

$$|W_i| = m = \left\lceil (p^2-1)\theta n\right\rceil.$$

We shall prove that the set $W=\bigcup_{i=2}^p W_i$ induces at least

$$\frac{1}{(p-1)^2}\left((p^2-1)\theta\right)^{p-1} n^{p-1}$$

$(p-1)$-cliques in R and thus obtain a contradiction with (1). The assertion is immediate for $p=2$; assume henceforth that $p\ge 3$. Let $w\in W$ be a vertex of minimum degree in $R[W]$, say let $w\in W_i$. We have

$$\begin{aligned}\delta(R[W]) &= \sum_{j\in[2..p]\setminus\{i\}} |\mathit{\Gamma}_R(w)\cap W_j| \ge \sum_{j\in[2..p]\setminus\{i\}} |\mathit{\Gamma}_R(w)\cap U_j| + |W_j| - |U_j|\\ &> (p-2)\left((1-\theta)\,n + m - n\right) = (p-2)\,(m-\theta n)\\ &\ge (p-2)\left(1-\frac{1}{p^2-1}\right)m.\end{aligned}$$

Hence, in view of $|W|=(p-1)m$,

$$\delta(R[W]) > \frac{p-2}{p-1}\left(1-\frac{1}{p^2-1}\right)|W| = \left(\frac{p-3}{p-2}+\frac{4p-5}{(p-1)^2(p+1)(p-2)}\right)|W|.$$

Applying Fact 3.25 to $R[W]$, we obtain

$$\begin{aligned}js_{p+1}(R) &\ge k_{p-1}(R[W]) > \frac{4p-5}{(p-1)^2(p+1)(p-2)}\cdot\frac{(p-2)^2}{(p-1)}\left(\frac{|W|}{p-1}\right)^{p-1}\\ &\ge \frac{1}{(p-1)^2}\left(\frac{|W|}{p-1}\right)^{p-1} \ge \frac{1}{(p-1)^2}\left((p^2-1)\theta\right)^{p-1} n^{p-1} > cn^{p-2},\end{aligned}$$

a contradiction with (1).

Hence, for every $u\in Y$, there exists two distinct color classes U_i and U_j such that (25) holds. For every $i\in[p]$, set

$$Z_i = \{u\colon u \in Z,\ \Gamma_R(u) \cap U_i = \varnothing\} \tag{26}$$

$$Y_i = \left\{u\colon u \in Y,\ \Gamma_R(u) \cap U_i \leq p^2\theta\right\}. \tag{27}$$

We have

$$\sum_{i=1}^{p} |Z_i| \geq \left|\bigcup_{i=1}^{p} Z_i\right| = |Z|, \quad \text{and} \quad \sum_{i=1}^{p} |Y_i| \geq 2\left|\bigcup_{i=1}^{p} Y_i\right| = 2|Y|.$$

Hence

$$\sum_{i=1}^{p}(|U_i| + |Z_i| + |Y_i|) \geq N + |Y| = p(n-1) + 1 + |Y|,$$

and there exists $i\in[p]$ such that

$$|U_i| + |Z_i| + |Y_i| \geq n - 1 + \left\lceil \frac{|Y_i| + 1}{p} \right\rceil.$$

Set $M=U_i$, $A=Z_i$, $C=Y_i$ and apply the following procedure to the sets A and C.

While $|M|+|A|+|C|>n-1+\lceil(|C|+1)/p\rceil$ *do*
if $C\neq\varnothing$ *remove a vertex from* C *else remove a vertex from* A;
end.

This procedure is defined correctly in view of $|M| = s$ and inequalities (24). Upon the end of the procedure we have

$$|M| + |A| + |C| = n - 1 + \left\lceil \frac{|C| + 1}{p} \right\rceil,$$

so condition (15) holds. We also see that

$$|A| = n - 1 + \left\lceil \frac{|C| + 1}{p} \right\rceil - |M| - |C| \leq n - |M| < \theta n,$$

so condition (16) holds as well. Finally, condition (17) holds due to

$$\frac{1}{2}|C| \leq \frac{p-1}{p}|C| = |C| - \frac{|C|+1}{p} - \frac{p-1}{p} + 1 \leq |C| - \left\lceil \frac{|C|+1}{p} \right\rceil + 1$$
$$\leq n - |M| < \theta n.$$

To complete the proof of the claim, observe that property (i) holds since the set M is independent in R; properties (ii) and (iii) hold in view of (26) and (27). ∎

Proof of Claim 3.16. Define a set $M'\subset M$ by

$$M'=\left\{u\colon u\in M,\ \ |\Gamma_R(u)\cap C|\geq\left(1-2p^2\theta\right)|C|\right\};$$

first we shall prove that $|M'|\geq|M|/2$. This is obvious if $C=\varnothing$, so we shall assume that $|C|>0$. We have

$$\begin{aligned}\left(1-p^2\theta\right)|C||M| &\leq \sum_{u\in C}|\Gamma_B(u)\cap M| = e_B(M,C)=\sum_{u\in M}|\Gamma_B(u)\cap C|\\ &=\sum_{u\in M'}|\Gamma_B(u)\cap C|+\sum_{u\in M\setminus M'}|\Gamma_B(u)\cap C|\\ &\leq |C||M'|+\left(1-2p^2\theta\right)|C|\left(|M|-|M'|\right),\end{aligned}$$

implying that

$$\left(1-p^2\theta\right)|M|\leq|M'|+\left(1-2p^2\theta\right)\left(|M|-|M'|\right)=\left(1-2p^2\theta\right)|M|+2p^2\theta|M'|,$$

and the desired inequality follows.

Setting

$$W_0=\{u\colon u\in V(H),\ d(u)\leq 2q\}\,,$$

by Proposition 3.20 we have $|W_0|\geq n/(2q+1)$. Since, by Fact 3.19, $H[W_0]$ is $(q+1)$-partite, there exists an independent set $W_1\subset W_0$ with

$$|W_1|\geq\frac{|W_0|}{q+1}\geq\frac{n}{(q+1)(2q+1)}>\theta n.$$

If $|A|+|M|\geq n$, we map H into $M\cup A$ as follows:

– select a set $W\subset W_1$ with $|W|=|A|$ – this is possible since $|A|<\theta n$;
– map arbitrarily W into A;
– map arbitrarily $V(H)\setminus W$ into M.

This mapping is a monomorphism since the set W is independent in H, the set M induces a complete graph in B, and the sets A and M induce a complete bipartite graph in B.

We assume henceforth that $|M|+|A|<n$. Since

$$|C|<2\theta n\leq\frac{n}{(q+1)(2q+1)},$$

select an independent set $W\subset W_1$ satisfying

$$|W|=n-|A|-|M|=|C|+1-\left\lceil\frac{|C|+1}{p}\right\rceil,$$

and set $P=\bigcup_{u\in W}\Gamma_H(u)$. Clearly

$$|P| \leq \sum_{u\in W}|\Gamma_H(u)| \leq 2q|W| \leq 2q|C|n \leq \frac{4q\theta}{1-\theta}|M| \leq \frac{|M|}{2} \leq |M'|.$$

We construct a monomorphism φ: $H\rightarrow B$ in two steps: *(a)* define φ on $H[W\cup P]$; *(b)* extend φ over $H-W-P$.

(a) **defining a monomorphism** φ: $H[W\cup P]\rightarrow B$

Define φ as an arbitrary one-to-one mapping φ: $P\rightarrow M'$ and extend φ by mapping W into C one vertex at a time. Suppose $W'\subset W$ is the set of vertices already mapped; if $W'\neq W$, select an unmapped $u\in W$ and let

$$\{v_1,\ldots,v_r\} = \varphi\left(\Gamma_H(u)\right) \subset M'.$$

Since $W\subset W_1\subset W_0$, we see that $r\leq 2q$. Then

$$\begin{aligned}\left|\bigcap_{i=1}^{r}\left(\Gamma_B(v_i)\cap C\right)\right| &\geq \sum_{i=1}^{r}|\Gamma_B(v_i)\cap C| - (r-1)|C| \\ &\geq r\left\lceil\left(1-2p^2\theta\right)|C|\right\rceil - (r-1)|C| \\ &\geq |C| + r\left\lceil -2p^2\theta|C|\right\rceil \geq |C| + \left\lceil -4p^2q\theta|C|\right\rceil \\ &\geq |C| + \left\lceil -\frac{|C|}{p}\right\rceil = |C|+1-\left\lceil\frac{|C|+1}{p}\right\rceil = |W| > |W'|.\end{aligned}$$

Hence, there exists a vertex $v\in\left(\bigcap_{i=1}^{r}(\Gamma_B(v_i)\cap C)\right)\backslash\varphi(W')$. Letting $v=\varphi(u)$, we extend φ to $W'\cup\{u\}$; this extension can be continued until the entire W is mapped into C.

(b) **extending** φ **over** $H-W-P$

Since $H-W-P$ is $(q+1)$-partite, the set $V(H)\setminus(W\cup P)$ contains an independent set W'' with

$$|W''| \geq \frac{n-|W|-|P|}{q+1} \geq \frac{n-(2q+1)|W|}{q+1} \geq \frac{1-(2q+1)\theta}{q+1}n \geq \theta n > |A|.$$

Now, extend φ to H by mapping arbitrarily W'' into A and $V(H)\setminus(W\cup P\cup W'')$ into $M\setminus\varphi(P)$. This extension is a monomorphism due to the following facts:

- W'' is independent in H,
- the set $E_H(W,W'')$ is empty,
- the set M induces a complete graph in B,
- the sets A and M induce a complete bipartite graph in B.

This completes the proof of the claim. ∎

3.2. Proof of Theorem 2.2

The proof of Theorem 2.2 is reduced to the following proposition.

Proposition 3.26. *For every $p \geq 3$, $c > 0$, there exists $b > 0$ such that if $G = G(n)$ is a graph with $js_p(G) > cn^{p-2}$, then $K_p(1,1,t,\dots,t) \subset G$, for $t > b\log n$.* ∎

In turn, Proposition 3.26 is implied by the following fact.

Fact 3.27. *For every $p \geq 3$, $c > 0$ there exists $b > 0$ such that if $G = G(n)$ is a graph with $k_p(G) \geq cn^p$, then $K_p(t) \subset G$ for $t \geq b\log n$.* ∎

The proof of this theorem can be found in [23].

3.3. Proof of Theorem 2.3

Lemma 3.28. *For every $p \geq 2$, $d \geq 1$ and $c > 0$, there exists $\alpha > 0$, such that if $G = G(n)$ and $k_p(G) > cn^p$, then G contains every p-partite graph H with $|H| \leq \alpha n$ and $\Delta(H) \leq d$.*

Proof. We sketch a proof using the Blow-up Lemma, see [20]. Applying the Regularity Lemma of Szemerédi we first find an ε-regular partition $V(G) = \bigcup_{i=0}^{k} V_i$ with $\varepsilon \ll (p,c)$, $1/\varepsilon \leq k \leq K(\varepsilon)$. Remove the vertices from V_0 and all edges that belong to:

- any $E(V_i)$;
- any irregular pair (V_i, V_j);
- any pair (V_i, V_j) with $\sigma_G(V_i, V_j) < c$.

A straightforward counting shows that the remaining graph contains a K_p, and so there exists p sets $V_{i_1}, \dots, V_{i_p}$ such that, for every $1 \leq l < j \leq p$, the pair (V_{i_l}, V_{i_j}) is ε-regular and $\sigma(V_{i_l}, V_{i_j}) > c$. Using Fact 3.3, we find subsets $U_{i_j} \subset V_{i_j}$ such that

- $|U_{i_1}| = \cdots = |U_{i_p}| \geq (1 - p\varepsilon)|V_{i_1}|$,
- for every $1 \leq l < j \leq p$, the pair (U_{i_l}, U_{i_j}) is 2ε-regular and every vertex $u \in U_{i_l}$ has at least $c/2$ neighbors in U_{i_j}.

According to the Blow-up Lemma, the graph $G\left[\bigcup_{j=1}^{p} U_{i_j}\right]$ contains all spanning graphs with maximum degree at most d, for $|U_{i_1}|$ sufficiently large. Therefore, $G\left[\bigcup_{j=1}^{p} U_{i_j}\right]$ contains all p-partite graphs of order $|U_{i_1}| + p - 1$ and of maximum degree at most d. Since $|U_{i_1}| > n/(2K)$, the assertion follows. ∎

3.4. Probabilistic Lemmas

We deduce Lemma 3.5 from a more general result; its proof is an adaptation of Sudakov's proof of Lemma 2.1 in [30].

Lemma 3.29. *Suppose G is a bipartite graph with parts V and U with $|V| = n$, $|U| = m$, and $e(G) \geq dnm$. Let H be a uniform k-graph with $V(H) = V$ and $d(v_1, \ldots, v_k) \leq am$ for every $\{v_1, \ldots, v_k\} \in E(H)$. Then there exists $W \subset V$ with $|W| \geq (d^i/2)n$ such that $e(H[W]) \leq (a/d)^i n^{k-1}|W|$.*

Proof. Chose $I \in U^i$ uniformly. Let $W = \Gamma(I)$ and define the random variables

$$X = |W|, \quad Y = e(H[W]), \quad Z = X - \frac{d^i}{a^i n^{k-1}} Y - \frac{d^i}{2} n.$$

We have

$$\mathbb{E}(X) = \frac{1}{m^i} \sum_{v \in V} d^i(v) \geq \frac{n}{m^i} \left(\sum_{v \in V} \frac{d(v)}{n} \right)^i \geq \frac{n}{m^i} (dm)^i = d^i n,$$

$$\mathbb{E}(Y) \leq \frac{1}{m^i} \sum_{\{v_1,\ldots,v_k\} \in E(H)} d^i(v_1, \ldots, v_k) \leq \frac{1}{m^i} e(H)(am)^i \leq a^i \frac{n^k}{2},$$

$$\mathbb{E}(Z) = \mathbb{E}(X) - \frac{d^i}{a^i n^{k-1}} \mathbb{E}(Y) - \frac{d^i}{2} n \geq \frac{d^i}{2} n - \frac{d^i}{a^i n^{k-1}} a^i \frac{n^k}{2} = 0.$$

Thus, there exists $I_0 \in U^i$ for which $\mathbb{E}(Z) \geq 0$. Then for $W = \Gamma(I_0)$ we have

$$|W| - \frac{d^i}{2} n = X - \frac{d^i}{2} n = Z + \frac{d^i}{a^i n^{k-1}} Y \geq 0,$$

$$e(H[W]) = Y = \frac{a^i n^{k-1}}{d^i} (X - Z) \leq \left(\frac{a}{d} \right)^i n^{k-1} |W|,$$

completing the proof. ∎

Proof of Lemma 3.5. Set $a = d^{2k/\lambda+1}$ and $n = |U_1|$; let i be the smallest integer such that $(a/d)^i n^k < 1$, i.e.,

$$i - 1 < \frac{k}{\ln(d/a)} \ln n = \frac{-\lambda}{2 \ln d} \ln n.$$

Define a k-uniform graph H with $V(H) = U_1$: a k-set $\{u_1, \ldots, u_k\} \subset U_1$ belongs to $E(H)$ if $d(u_1, \ldots, u_k) \leq a|U_2|$. According to Lemma 3.29, there exists $W \subset U_1$ with $|W| \geq (d^i/2)n$ and

$$e(H[W]) \leq \left(\frac{a}{d} \right)^i n^{k-1} |W| \leq \left(\frac{a}{d} \right)^i n^k < 1.$$

Thus, W is an independent set in H, and so $d(u_1,\dots,u_k) > a|U_2|$ for every k-set $\{u_1,\dots,u_k\} \subset W$. We also have, for n large,

$$|W| \geq \frac{d^i}{2} n \geq \frac{d}{2} n^{1-\lambda/2} > n^{1-\lambda},$$

completing the proof. ∎

4. Degenerate and splittable graphs

Proposition 2.8 follows from the corollary to the following lemma.

Lemma 4.1. *Let $k \geq 1, n \geq 2$ be integers. For any tree T_n of order n, there exists a set $S_k \subset V(T_n)$ such that $|S_k| \leq 2^{k+2} - 6$ and $\psi(T_n - S_k) \leq 2^{-k} n$.*

Proof. We shall use induction on k. According to a result from [12], either $\psi(T_n - uv) \leq 2n/3$ for some $uv \in E(T_n)$, or $\psi(T_n - u) \leq n/3$ for some $u \in V(T_n)$. Therefore, $\psi(T_n - u - v) \leq n/2$ for some vertices $u, v \in V(T_n)$, implying the lemma for $k = 1$ with $S_1 = \{u, v\}$. Assume the lemma holds for $k-1$ and let S_{k-1} be a set such that $\psi(T_n - S_{k-1}) \leq 2^{-k+1} n$. For each component C of $T_n - S_{k-1}$ with $|C| > 2^{-k} n$, select two vertices $u, v \in V(C)$ such $\psi(C - u - v) \leq |C|/2 \leq 2^{-k} n$. Since there are fewer than 2^k components C satisfying $|C| > 2^{-k} n$, we deduce that $|S_k| < |S_{k-1}| + 2^{k+1}$, completing the induction step and the proof. ∎

Corollary 4.2. *Suppose $0 < \gamma < 1$ is fixed. For every $0 < \eta < 1$, every sufficiently large tree is (γ, η)-splittable.*

Proof. Set $k = \lceil \log_2 1/\varepsilon \rceil$. Lemma 4.1 implies that there exists $S \subset V(T_n)$ such that $|S| < 2^{k+2} - 6$ and $\psi(T_n - S) \leq 2^{-k} n \leq \eta n$. We deduce that $|S| < 2^{k+2} - 6 < 2^{k+2} < 8\eta^{-1} < n^{1-\gamma}$ for n large. ∎

Next we sketch the proofs of Proposition 2.6 and 2.5.

Proof of Proposition 2.6. If $\Delta(G) \leq q$ then $\Delta(G^k) \leq q^k$; hence $\mathcal{F}^k$ is degenerate. Let $\mathcal{F}$ be γ-crumbling, $G \in \mathcal{F}$ is a graph of order n and $M \subset V(G)$ is a set such that $|M| < n^{1-\gamma}$ and $\psi(G - M) < \varepsilon n$. Set

$$\left\{ M' = v \colon v \in V(G), \text{ there exists } u \in M \text{ with } \operatorname{dist}(u, v) \leq k \right\}.$$

If A and B are components of $G - M$, then $\operatorname{dist}(A - M', B - M') \geq 2k$. Therefore, $\psi(G^k - M') < \varepsilon n$, implying that $\mathcal{F}^k$ is $(\gamma/2)$-crumbling. ∎

Proof of Proposition 2.5. Burr and Erdős [7, Lemma 5.4] proved that for every graph G there exists $k\geq 1$ such that every graph of order n homeomorphic to G can be embedded in P_n^k. This completes the proof, in view of Propositions 2.7 and 2.6. ∎

Proof of Proposition 2.10. Observe that if G_1 is q_1-degenerate and G_2 is q_2-degenerate then $G_1\times G_2$ is (q_1+q_2)-degenerate. Also let $G_1=G(n)$ be a (γ_1,η_1)-splittable graph and $G_2=G(m)$ be a (γ_2,η_2)-splittable graph. Suppose $m\leq n$, select $M\subset V(G_1)$ with $|M|<n^{1-\gamma_2}$ such that $\psi(G_1-M)<\eta_1 n$. Then

$$|M\times V(G_2)| = n^{1-\gamma_1}m \leq (mn)^{1-\gamma_1/2}$$

and $\psi(G_1\times G_2-M\times V(G_2))<\eta_1 nm$. Therefore, the graph $G_1\times G_2$ is (γ,η)-splittable with $\gamma=\min\{\gamma_1/2,\gamma_2/2\}$ and $\eta=\max\{\eta_1,\eta_2\}$. ∎

Proof of Proposition 2.11. Let $\mathcal{F}$ be a γ-crumbling family. Suppose $G\in\mathcal{F}$ is a graph of order n and $M\subset V(G)$ is such that $|M|<n^{1-\gamma}$ and $\psi(G-M)<\eta n$. Let $\varphi\colon G^{\mathbf{k}_n}\to G$ be the homomorphism mapping every vertex to its ancestor. From the graph $G^{\mathbf{k}_n}$ remove the set $M'=\varphi^{-1}(M)$. If C is a component of $G-M$, then $\varphi^{-1}(C)$ is a component of $G^{\mathbf{k}_n}-M'$ and so

$$\psi\big(G^{\mathbf{k}_n}-M'\big)\leq K\psi(G-M)<K\eta n.$$

Also,

$$|M'|\leq K|M|<Kn^{1-\gamma}<(Kn)^{1-\gamma/2}$$

for n large. Hence, $\{G^{\mathbf{k}_n}\}$ is a $(\gamma/2)$-crumbling family. ∎

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